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Abstract. Radiative transfer is a phenomenon that has an importance in a wide range of applications from climatology to astrophysics. Depending on the physical regimes involved, a hierarchy of models may be used, each of which having drawbacks and qualities. However, there are still applications for which no model is fully satisfying. An example is ICF (Inertial Confinement Fusion) where dozens of lasers converge on a fuel pellet of the size of a pinehead. This kind of simulation requires a coupling between radiation and other processes and hence one would require a model that is cheap enough in terms of computation cost to carry it out. The $M1$ model [3] may then seem to be an interesting choice. But the directional complexity of the problem and the fact that the energy is mainly located inside a narrow frequency interval is hardly compatible with this model and one would rather use an (expensive) model such as a kinetic model.
In this paper, we introduce an hybrid model that mixes the multigroup-$M1$ model and a kinetic model. This hybrid model intends to be an extension of both of them and therefore adds degrees of freedom that allows to properly take into account a wide variety of problems. We also show that it possesses important properties including the correct asymptotic limit in the diffusion regime and the local decreasing of the total entropy.

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Introduction

The radiative transfer equation (RTE) describes the evolution of the radiative intensity $I_\nu(\Omega) = I(t, x, \Omega, \nu)$ where Ω is the photons' direction of propagation and ν is the frequency. This radiative intensity $I_\nu(\Omega)$ is linked to the photons' distribution function. Assuming local thermal equilibrium and neglecting scattering, the RTE writes:

$$\frac{1}{c}\partial_t I_\nu(\Omega) + \Omega.\nabla I_\nu(\Omega) = \sigma_\nu\Big(B_\nu(T) - I_\nu(\Omega)\Big), \tag{0.1}$$

where c is the speed of the light, σ_ν is the emission opacity, T is the material temperature and $B_\nu(T)$ is Planck's blackbody function given by:

$$B_\nu(\Omega) = \frac{2h\nu^3}{c^2}\Big[\exp\Big(\frac{h\nu}{kT}\Big) - 1\Big]^{-1}. \tag{0.2}$$

The constants h and k are respectively Planck and Boltzmann constants. It is to note that the energy of B (ie its integral over all directions and frequencies) is proportional to T^4. The constant of proportionality is denoted by $a(\simeq 7.56.10^{-16}\ SI)$.
Due to emission and absorption process, the radiative energy is not conserved. To preserve the conservation of

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the total (radiative+material) energy, the following simplified temperature evolution equation will be used in this paper:

$$\rho C_v \partial_t T = \int_0^\infty \int_{S^2} \sigma_\nu \Big(I_\nu(\Omega) - B_\nu(T) \Big) d\Omega d\nu. \tag{0.3}$$

The direction of propagation Ω and the frequency ν play a very different role in the RTE. Indeed, the opacity σ_ν is a critical parameter. When $\sigma_\nu = 0$ the RTE is nothing but a convection equation and the associated regime is called the transport (or free-streaming) regime. On the other hand, when $\sigma_\nu \to \infty$, the RTE degenerates into the *diffusion* equation (see [5], [11] and [8]):

$$\partial_t(\rho C_v T + 4aT^4) - \nabla_x \Big(\frac{4acT^3}{3\sigma} \nabla_x T \Big) = 0. \tag{0.4}$$

A lot of simulations involve free-streaming, diffusion as well as in-between regimes altogether. It is therefore crucial to insure that the models we use correctly handle all regimes.
Moreover, σ_ν has huge frequencial variations due to phenomena such as bound-bound transitions which produce emission lines that are very narrow. Thus *grey* models (ie integrated over all frequencies) are often too coarse and introduce significant errors due to frequency averaging. The models we use hence have several frequency degrees of freedom. The frequencies are split into *groups* and *bands* respectively indexed by q and k:

$$[0, +\infty[_\nu = \Big(\bigcup_{q=1}^{Q} \mathcal{Q}_q \Big) \bigcup \Big(\bigcup_{k=1}^{K} \overline{\mathcal{Q}}_k \Big).$$

As a notation and to avoid confusion, every frequency group in this paper will be indexed by q and every frequency band will be indexed by k. Inside the bands $\overline{\mathcal{Q}}_k$, a quadrature formula is used to approximate integrals over directions and frequencies. The following notations will be used throughout this paper:

$$\begin{aligned}
&< \bullet >_q = \frac{1}{c} \int_{\mathcal{Q}_q} \int_{S^2} \bullet \, d\Omega d\nu, \\
&\overline{< \bullet >} = \sum_{k=1}^{K} \sum_{l=1}^{L} \bullet \, \omega_l \xi_k \ \simeq \frac{1}{c} \int_{\cup \overline{\mathcal{Q}}} \int_{S^2} \bullet \, d\Omega d\nu, \\
&< \bullet > = \sum_{q=1}^{Q} < \bullet >_q + \overline{< \bullet >}.
\end{aligned}$$

Throughout this paper, frequency and direction indexes will be omitted inside integrals and quadrature formulas except when necessary.

- *Bands* refer to frequency intervals that are small enough to make several assumptions. For instance, Planck's function may be supposed to be constant inside a band. Among others, these assumptions allow to use several useful techniques to compute mean values of σ_ν (eg CK methods). A typical simulation may take into account thousands of frequency bands.
- *Groups* refer to large frequency intervals where the previous assumptions cannot be made. A particular attention must then be paid to the way to compute mean values of σ_ν. A typical multigroup simulation may consider a few frequency groups (up to several dozens).

The computation of the opacities' mean values inside groups may be problematic for modelling issues as we will see later. Among the classical choices to do so are Planck and Rosseland mean values, respectively denoted by

σ_P and σ_R and given by:

$$\sigma_P = \sigma_P(T) = \frac{<\sigma B(T)>}{<B(T)>}, \tag{0.5}$$

$$\sigma_R = \sigma_R(T) = \frac{<\partial_T B(T)>}{<\frac{1}{\sigma}\partial_T B(T)>}. \tag{0.6}$$

A lot of physical applications involve radiation in a wide range of regimes. It is possible to distinguish between two categories considering the influence of radiation on the flow:

- The radiation impact on the flow is neglectable. This is the case for instance of climatology applications.
- The radiation effects have an impact on the flow. This happens in superorbital atmospheric reentry, astrophysics (radiative shocks,...) and ICF (fusion obtained by impacting laser beams on a deuterium/tritium target).

These two categories imply very different levels of modelling. For the latter, a full coupling between hydrodynamics and radiation may be mandatory and requires the use of a robust and cheap radiative model.

The objective of this paper is to introduce an hybrid model which mixes a kinetic and a moments model. This hybrid model shall be supple enough to naturally adapt to any simulation, even though the main target applications requires a full coupling between radiation and material and a specific geometry (eg convergence of lasers for ICF).
This article will be organized as follows. In the next section, we will briefly recall a kinetic model suitable for a narrow-band level of frequencial resolution. We will mainly focus on its construction and its main properties. Then a section will be dedicated to the $M1$ model. The multigroup version will be recalled, we will also see that it has several important properties that make it robust and a very interesting candidate for multigroup simulations. We will focus on one of its most essential characteristic which is the possibility to consistantly handle the opacities' mean values.
Finally, we will introduce the way of coupling these two models inside an hybrid model. We will then show that this model still preserves two fundamental properties: the asymptotic behaviour in the diffusion regime and the local decreasing of the total entropy. A simple numerical example will illustrate its interest.

1. A kinetic model ($Q = 0$)

This model was first introduced in [1]. It is designed for using frequency bands. One of the consequences of this choice is that the mean values of the opacity σ_k do not depend on variations of B or I. Therefore, σ_k has the same value that Planck (0.5) or Rosseland (0.6) mean values. Aside from these frequency bands, we also consider a finite number of directions $(\Omega_l)_{l=1\dots L}$ and look for an approximation of $I_{\nu_k}(\Omega_l) = I(t, x, \Omega_l, \nu_k)$ denoted $J_{k,l}$ which is solution of the following discrete system:

$$\frac{1}{c}\partial_t J_{k,l} + \Omega_l \nabla J_{k,l} = \sigma_k \big[\mathcal{B}_k(T) - J_{k,l}\big],\ (k,l) \in [1\dots K] \times [1\dots L] \tag{1.1}$$

$$\rho C_v \partial_t T = \overline{<c\sigma(J - \mathcal{B}(T))>}. \tag{1.2}$$

The main difficulty is to carefully define the discrete equilibrium function $\mathcal{B}_k(T)$. The "naive" choice $\mathcal{B}_k(T) = B_{\nu_k}(T)$ is not sufficient enough to insure the conservation of several important physical properties. For example, it is easy to see that in general $\sum_{k=1}^{K}\sum_{l=1}^{L} B_{\nu_k}(T)\ \omega_l \xi_k \neq aT^4$. Therefore, some energy is artificially created (or lost) only due to this form of the discrete equilibrium function.
A solution is to use Levermore's technique of minimum entropy principle [7] in a fashion related to Mieussens' choice for Boltzmann-BGK [9]: the discrete equilibrium function is chosen to be the minimum of the discrete

radiative entropy:

$$\mathcal{B} = (\mathcal{B}_{k,l}(T))_{k,l} = \text{argmin}\{\overline{< h_d(J) >},\ \overline{< J >} = aT^4\}, \tag{1.3}$$
$$h_d(J) = \frac{2k\nu_k^2}{c^3}\Big[n_{k,l}\ln(n_{k,l}) - (n_{k,l}+1)\ln(n_{k,l}+1)\Big],$$
$$n_{k,l} = n(J)_{k,l} = \frac{c^2}{2h\nu_k^3}J_{k,l}.$$

This choice is consistent in the sense that Planck's function is the minimum of the radiative entropy among all the functions whose energy is aT^4. Therefore $(\mathcal{B}_{k,l})_{k,l}$ can be considered as an approximation of B_ν.

Theorem 1.1. *If $T > 0$ then there exists a unique $\mathcal{B}$ such that:*

$$\mathcal{B}_{k,l} = \mathcal{B}_k = \frac{2h\nu_k^3}{c^2}\Big[\exp\Big(\frac{h\nu_k\alpha}{k}\Big) - 1\Big]^{-1}, \tag{1.4}$$

where α is the unique solution of $\overline{< \mathcal{B} >} = aT^4$.

Proof. This results comes from a straightforward computation, which is briefly (and formally) recalled here. Let us introduce the Lagrangian of the problem (1.3):

$$\mathcal{L}(J,\lambda) = \overline{< h_d(J) >} - \lambda(aT^4 - \overline{< J >}). \tag{1.5}$$

According to Lagrange's theorem, if $\mathcal{B}$ is a solution of (1.3) then there exists $\alpha \in \mathbb{R}$ such that $(\mathcal{B}, \alpha)$ is a saddle-point of $\mathcal{L}$. Moreover, thanks to the convexity of the discrete entropy, we can say that if $(\mathcal{B}, \alpha)$ is a saddle-point of $\mathcal{L}$, then it is a solution of (1.3). At the saddle-points of $\mathcal{L}$, if $\mathcal{L}$ is derivable, we have $\partial_J\mathcal{L}(\mathcal{B}, \alpha) = 0$ hence:

$$\forall\varphi \in \mathcal{D}(S^2 \times \mathbb{R}^+),\ < \overline{< h_d'(J) >}, \varphi >_{\mathcal{D}',\mathcal{D}} = - < \overline{< \alpha >}, \varphi >_{\mathcal{D}',\mathcal{D}} .$$

In particular, the last equality stands for all $\varphi \in \mathcal{D}(S_l^2 \times \overline{\mathcal{Q}}_k)$, where S_l^2 is the part of the unit sphere around the direction Ω_l. Thus we can say that:

$$\forall\ k,l,\ h_d^{'}(\mathcal{B}_{k,l}) = -\alpha,$$
$$\frac{k}{h\nu_k}\ln(1 + \frac{1}{n(\mathcal{B})_{k,l}}) = -\alpha,$$

hence

$$\mathcal{B}_{k,l} = \frac{2h\nu_k^3}{c^2}\Big[\exp\Big(\frac{h\nu_k\alpha}{k}\Big) - 1\Big]^{-1}.$$

□

There are two points that have to be emphasized here. First, the discrete equilibrium function -as Planck's function- is isotropic: $\mathcal{B}_{k,l} = \mathcal{B}_k$. Among others, a consequence of this is that only the sum over k is required to compute α.
Lagrange's multiplier $\alpha = \alpha(T)$ plays an important role. It is a scalar regardless of the number of frequency bands considered and its determination requires to solve the nonlinear scalar equation $\overline{< \mathcal{B}(\alpha) >} = aT^4$.
If the quadrature formula was exact, then we would have $\alpha = 1/T$. Using $1/T$ as a starting value for any suitable efficient numerical method allows to converge quickly, hence the determination of α does not involve expensive computations.

Proposition 1.2. *The system* (1.1)-(1.2) *is hyperbolic symmetrizable. Moreover, if the quadrature formula* $\overline{<\bullet>}$ *is wisely chosen then it has the following properties:*

- *the total energy is conserved:*

$$\partial_t(\overline{<J>} + \rho C_v T) + c\ \nabla_x \overline{<\Omega_l J>} = 0,$$

- *if* $J(t=0) \in L^\infty$ *then* $\forall t > 0,\ J(t) \in L^\infty$,
- *the total entropy is locally decreasing:*

$$\partial_t \left(\frac{\overline{<h(J)>}}{c} + \rho C_v \mathcal{U} \right) + \nabla_x \overline{<\Omega_l h_d(J)>} \leq 0.$$

  where $\mathcal{U} = -\int_0^T \alpha(T) dT$ *is a convex function of* T.
- *it is asymptotic preserving in the diffusive limit.*

The first two results above were proved in previous papers (see [1] and [15]). The last two will be proved for the more general hybrid model -which includes this kinetic model- later in this article.

Unfortunately, the number of bands required for any kinetic model implies that these models are way to expensive in a fully coupled context involving radiation, hydrodynamics and other processes such as chemical reactions or turbulence.

2. Multigroup-$M1$ model ($K = 0$)

Since our main objective is to perform fully coupled simulations, we have to consider cheaper models ie models that are somehow integrated over directions and frequencies. There are two categories of such models: flux-limited diffusion and moments models (see [12] and references therein for a list of the classic choices). Our choice is to select the $M1$ model, which belongs to the category of moments models. It was introduced in [2] and has several variations among which [14], [3], [13] and [4]. To built it, the first step consists in obtaining the moments equations from (0.1). To do so, an integration over directions and inside each frequency group $\mathcal{Q}_q$ is performed for (0.1) and $\Omega \times$ (0.1). It leads to the following system:

$$q = 1 \dots Q,\ \partial_t E_q + \nabla F_q = \sigma_q^e a \theta_q^4 - \sigma_q^a E_q, \tag{2.1}$$

$$\partial_t F_q + c^2 \nabla P_q = -\sigma_q^f F_q. \tag{2.2}$$

The radiative energy E_q, the radiative flux vector F_q and the radiative pressure tensor P_q are the first three moments of the radiative intensity inside the q^{th} frequency group. Moreover, $\sigma^e,\ \sigma^a$ and σ^f are mean values of the opacity and their choice will be discussed later. These variables are defined as:

$$
\begin{aligned}
E_q &= < I >_q, \\
F_q &= < c\Omega I >_q, \\
P_q &= < \Omega \otimes \Omega I >_q, \\
a\theta_q^4 &= < B(T) >_q, \\
\sigma_q^e &= \frac{< \sigma B(T) >_q}{a\theta_q^4}, \qquad (2.3) \\
\sigma_q^a &= \frac{< \sigma I >_q}{E_q}, \qquad (2.4) \\
\sigma_q^f &= \begin{pmatrix} \dfrac{< \sigma \Omega^x I >_q}{F_q^x} & 0 \\ 0 & \dfrac{< \sigma \Omega^y I >_q}{F_q^y} \end{pmatrix}. \qquad (2.5)
\end{aligned}
$$

The system (2.1)-(2.2) is the generic multigroup moments system. It is not closed. To close it, one has to make an hypothesis on the radiative pressure P_q so that it can be expressed as a function of E_q and F_q. The choice of the closure determines the model. There are several ways to choose the closure, including the classic $P1$ model where $P_q := I_d E_q/3$. This shape is based on the fact that at the radiative equilibrium (ie whenever $I_\nu(\Omega) = B_\nu(T)$), the radiative pressure is indeed one third of the radiative energy times the identity tensor. The main drawback of the $P1$-model is that it does not predict physical solutions far from the radiative equilibrium.

Definition 2.1. A solution (E_q, F_q) will be said to be *physically relevant* if $E_q > 0$ and $\|F_q\| \leq cE_q$.

In fact, this definition is simply the consequence that E_q and F_q are the first two moments of some (positive) radiative intensity. The second property is called the (radiative) *flux limitation*. It is very important yet hard to preserve in models. For instance, the $P1$ model violates this property and thus sometimes predicts unphysical solutions.
There are several other options to close the system (2.1)-(2.2) (see [12] and references in [14]). One of which is to use a minimum entropy principle in a fashion similar to what Levermore did for fluid mechanics [6]. It is to note that other models based on a minimum entropy principle exist to close this system (see for example [10]), however the $M1$ model is based on the (physical) radiative entropy:

$$
\begin{aligned}
h(I) &= \frac{2k\nu^2}{c^3}\Big[n_I \ln(n_I) - (n_I+1)\ln(n_I+1)\Big], \qquad (2.6) \\
n_I &= \frac{c^2}{2h\nu^3} I_\nu(\Omega). \qquad (2.7)
\end{aligned}
$$

The first step is to build an underlying radiative intensity $\mathcal{I}$ thanks to the minimum entropy principle:

$$
\mathcal{I} = \operatorname{argmin}\{< h(I) > \ / \ \forall q = 1 \dots Q \ \ < I >_q = E_q, \ < c\Omega I >_q = F_q\}. \qquad (2.8)
$$

The solution of this minimization problem is:

$$
\mathcal{I} = \sum_q 1_q \frac{2h\nu^3}{c^2}\Big[\exp(\frac{h\nu}{kT}\alpha_q.(1,\Omega)^\top) - 1\Big]^{-1}, \qquad (2.9)
$$

where 1_q is the characteristic function of the q^{th} group and α_q is the Lagrange multiplier of the minimization problem (fully determined by the constraints).
The $M1$ form of the radiative pressure is simply the corresponding moment of the underlying radiative intensity:

$$P_q := < \Omega \otimes \Omega \mathcal{I} >_q .$$

P_q may be expressed in Eddington form as follows:

$$P_q = D_q E_q, \tag{2.10}$$

$$D_q = \frac{1-\chi_q}{2} I_d + \frac{3\chi_q - 1}{2} \frac{F_q \otimes F_q}{\|F_q\|}, \tag{2.11}$$

where the Eddington factor χ_q is the eigenvalue of D_q associated with F_q.

Proposition 2.2. *The system* (2.1)-(2.2) *with the* $M1$ *choice of the closure is hyperbolic symmetrizable. Moreover, it has the following properties:*

- *the total energy is conserved,*
- *the total entropy is locally decreasing,*
- *it is asymptotic preserving in the diffusive limit,*
- *it has a natural flux limitation inside each group (ie* $\|F_q\| \leq cE_q$*).*

Once again, these properties were proved in previous articles ([14], [15]) or will be proved for the more general hybrid model in the next section of this article.
One of the edges the $M1$ model has compared to other related models is the possibility to chose consistent mean opacities. As a matter of fact, the opacities' mean values required for moments or flux-limited diffusion models are problematic since their expression should be given by (2.3), (2.4) and (2.5). On the one hand, (2.3) is nothing but Planck's mean value of the opacity which is relatively easy to compute. But on the other hand, (2.4) and (2.5) are mean values relatively to functions of I, which is not reachable (the unknowns of our model are its moments). As a consequence, one has to make an additional assumption to compute these mean opacities. Classic choices include taking Planck, Rosseland or Chandrasekhar mean values but this additional assumption has a cost in terms of precision of the model.
Hopefully for the $M1$ model, even if the real radiative intensity cannot be used, we have built an underlying one $\mathcal{I}$ which may be considered to approximate (2.4) and (2.5). For instance:

$$\sigma_q^a \simeq \frac{< \sigma \mathcal{I} >_q}{E_q}. \tag{2.12}$$

Since $\mathcal{I}$ is fully determined by the knowledge of E_q and F_q, it is possible to make all the computations. This choice is consistent in the sense that no additional assumption is made. It allows to have a good precision even with a few number of groups: see the test-cases in [14].

Using the consistent mean values of the opacity, the multigroup-$M1$ model is equivalent to the following formulation:

$$< \frac{1}{c} \partial_t \mathcal{I} + \Omega \nabla \mathcal{I} >_q = < \sigma \Big[B(T) - \mathcal{I} \Big] >_q, \ q = 1 \dots Q \tag{2.13}$$

$$< \frac{1}{c} \partial_t (\Omega \mathcal{I}) + \nabla (\Omega \otimes \Omega \mathcal{I}) >_q = - < \sigma \Omega \mathcal{I} >_q, \tag{2.14}$$

$$\mathcal{I}_q = 1_q \mathcal{I}_q(E_q, F_q), \tag{2.15}$$

where $\mathcal{I}_q$ is the underlying radiative intensity used as a closure function for the $M1$ model (2.9).

The multigroup-$M1$ is usually a good choice for full coupling purposes. It is relatively cheap, very robust, conserves the fundamental physical properties and even has a natural way to approximate mean opacities. However, it has a directional drawback since it has to cope with only one direction of anisotropy per group (colinear to F_q): it cannot handle the case when at a given point two or more comparable energies converge from opposite directions (see [3] for an example). Depending on the geometry of the problem, this drawback may be problematic or not. For instance, for computing the fluxes on the front shield of a probe or to simulate an explosion, this directional effect has nearly no impact.
The $M1$ model has been declined into half and partial space $M1$ models [3], [13], [4]. For 1D simulations, the half-$M1$ model (ie integrated over half of the unit sphere for directions) nearly annihilate the directional effect, but for 2D and 3D simulations, even partial $M1$ might not be sufficient. An example of tricky simulation would be a ICF-type problem where dozens of lasers converge on 1 mm-radius zone.

3. Hybrid Model

As we have seen in previous sections, both the kinetic and the multigroup $M1$ models may have some drawbacks. The former is often too expensive and the latter sometimes has directional problems. Starting from the observation that these two models are complementary, we derive an hybrid model which mixes the multigroup-$M1$ model inside frequency groups and the kinetic model inside frequency bands:

$$\forall q = 1 \dots Q,\ \partial_t E_q + \nabla . F_q = c\big[\sigma_q^e a\theta_q^4 - \sigma_q^a E_q\big], \tag{3.1}$$

$$\partial_t F_q + c^2 \nabla P_q = -c\sigma_q^f F_q, \tag{3.2}$$

$$\forall k = 1 \dots K,\ \forall l = 1 \dots L,\ \frac{1}{c}\partial_t J_{k,l} + \Omega_l \nabla J_{k,l} = \sigma_k \big[\mathcal{B}_k(T) - J_{k,l}\big], \tag{3.3}$$

$$\rho C_v \partial_t (T - \mathcal{T}) = \sum_{q=1}^{Q} (\sigma_q^a E_q - \sigma_q^e a\theta_q^4) \tag{3.4}$$

$$\rho C_v \partial_t \mathcal{T} = \overline{< c\sigma(J - \mathcal{B}(T)) >} \tag{3.5}$$

$$\mathcal{I}_q = \text{argmin}\Big\{ < h(I) >_q \ / \forall q = 1 \dots Q,\ < I >_q = E_q \text{ and } < c\Omega I >_q = F_q \Big\} \tag{3.6}$$

$$\mathcal{B} = \text{argmin}\Big\{ \overline{< h_d(I) >} / \overline{< I >} = aT^4 - \sum_{q=1}^{Q} a\theta_q^4 \}. \tag{3.7}$$

Where $\mathcal{T} = \mathcal{T}(T)$ is such that:

$$\mathcal{T} = \Big(\frac{\overline{< \mathcal{B}(T) >}}{a} \Big)^{1/4}. \tag{3.8}$$

This model may be seen as an improvement of either the $M1$ or the kinetic model. Typically, groups may be chosen in regions were either very little energy exists or there is no directional problem whereas bands are chosen where a large amount of energy lies inside a few narrow bands (eg lasers) or may converge at a given point.

Of course, it is crucial to insure that this hybrid model behaves well. The conservation of the energy is straightforward and the next two theorems respectively show that it is still asymptotic preserving in the diffusive limit and locally decreases the total entropy.

Theorem 3.1. *If $(\overline{\mathcal{Q}})_k$ are narrow bands, σ_ν is constant inside each frequency group $\mathcal{Q}_q$ and if the quadrature formula $\overline{< \bullet >}$ is at least 2nd-order, then the model* (3.1) − (3.7) *is asymptotic preserving in the diffusive limit*

ie when $\sigma \to \infty$ it degenerates into the equation:

$$\partial_t(\rho C_v T + 4aT^4) - \nabla\Big(\frac{4acT^3}{3\sigma_R}\nabla_x T\Big) = 0, \tag{3.9}$$

where σ_R is Rosseland's mean value of the opacity given by (0.6).

Proof. Adding (3.4) and (3.5) together, and using the Eddington form of P_q (2.10), the system (3.1) – (3.7) may be written as:

$$\begin{aligned}
\forall q = 1 \dots Q,\ \partial_t E_q + \nabla.F_q &= c\big[\sigma_q a\theta_q^4 - \sigma_q E_q\big],\\
\partial_t F_q + c^2\nabla(D_q E_q) &= -c\sigma_q F_q,\\
< \frac{1}{c}\partial_t(\Omega\mathcal{I}) + \nabla(\Omega\otimes\Omega\mathcal{I}) >_q &= - < \sigma\Omega\mathcal{I} >_q,\\
\frac{1}{c}\partial_t J_{k,l} + \Omega_l\nabla J_{k,l} &= \sigma_k\big[\mathcal{B}_k(T) - J_{k,l}\big],\ k = 1\dots K,\ l = 1\dots L,\\
\rho C_v\partial_t T &= \sum_{q=1}^{Q}(\sigma_q E_q - \sigma_q a\theta_q^4) + \overline{< c\sigma(J - \mathcal{B}(T)) >}.
\end{aligned}$$

To exhibit the asymptotic limit of this model in the diffusion regime, a Chapmann-Enskog-like expansion is used. First, the equations are scaled setting:

$$\tilde{t} := \varepsilon t,\ \tilde{\sigma} := \varepsilon\sigma.$$

The system thus becomes:

$$\frac{\varepsilon^2}{c}\partial_{\tilde{t}} E_q + \varepsilon\nabla F_q = \tilde{\sigma}_q\Big[a\theta_q^4 - E_q\Big],\ q = 1\dots Q, \tag{3.10}$$

$$\frac{\varepsilon^2}{c}\partial_{\tilde{t}} F_q + \varepsilon\nabla(D_q E_q) = -\tilde{\sigma}_q F_q, \tag{3.11}$$

$$\frac{\varepsilon^2}{c}\partial_{\tilde{t}} J_{k,l} + \varepsilon\Omega_l\nabla J_{k,l} = \tilde{\sigma}_k\big[\mathcal{B}_k(T) - J_{k,l}\big],\ k = 1\dots K,\ l = 1\dots L, \tag{3.12}$$

$$\varepsilon^2\rho C_v\partial_{\tilde{t}} T = \sum_{q=1}^{Q}(\sigma_q E_q - \sigma_q a\theta_q^4) + \overline{< c\tilde{\sigma}(J - \mathcal{B}) >}. \tag{3.13}$$

Then an asymptotic expansion of E, F and J is performed:

$$\begin{aligned}
E &= E^0 + \varepsilon E^1 + \varepsilon^2 E^2 + \dots,\\
F &= F^0 + \varepsilon F^1 + \varepsilon^2 F^2 + \dots,\\
J &= J^0 + \varepsilon J^1 + \varepsilon J^2 + \dots.
\end{aligned}$$

The use of this expansion in (3.10)-(3.13) and an identification of the coefficients lead to:

$$\underline{\varepsilon^0 \text{ terms}} : E_q^0 = a\theta_q^4, \tag{3.14}$$
$$F_q^0 = 0, \tag{3.15}$$
$$J_{k,l}^0 = \mathcal{B}_k(T), \tag{3.16}$$
$$0 = \sum_{q=1}^{Q} (\tilde{\sigma}_q E_q^0 - \tilde{\sigma} a\theta_q^4 + \overline{\langle c\tilde{\sigma}(J^0 - \mathcal{B})\rangle}. \tag{3.17}$$
$$\underline{\varepsilon^1 \text{ terms}} : \nabla F_q^0 = -\tilde{\sigma}_q E_q^1, \tag{3.18}$$
$$\nabla (DE)_q^0 = -\tilde{\sigma}_q F_q^1, \tag{3.19}$$
$$\Omega_l \nabla J_{k,l}^0 = -\tilde{\sigma}_k J_{k,l}^1, \tag{3.20}$$
$$0 = \sum_{q=1}^{Q} \tilde{\sigma}_q E_q^1 + \overline{\langle c\tilde{\sigma} J^1\rangle}. \tag{3.21}$$
$$\underline{\varepsilon^2 \text{ terms}} : \frac{1}{c}\partial_{\tilde{t}} E_q^0 + \nabla F_q^1 = -\tilde{\sigma}_q E_q^2, \tag{3.22}$$
$$\frac{1}{c}\partial_{\tilde{t}} F_q^0 + \nabla (DE)_q^1 = -\tilde{\sigma}_q F_q^2, \tag{3.23}$$
$$\frac{1}{c}\partial_{\tilde{t}} J_{k,l}^0 + \Omega_l \nabla J_{k,l}^1 = -\tilde{\sigma}_k J_{k,l}^2, \tag{3.24}$$
$$\rho C_v \partial_{\tilde{t}} T = \sum_{q=1}^{Q} \tilde{\sigma}_q E_q^2 + \overline{\langle c\tilde{\sigma} J^2\rangle}. \tag{3.25}$$

The expression of F_q^1 and $J_{k,l}^1$ are respectively taken from (3.19) and (3.20):

$$F_q^1 = -\frac{1}{\tilde{\sigma}_q}\nabla (DE)_q^0, \tag{3.26}$$
$$J_{k,l}^1 = -\frac{1}{\tilde{\sigma}_k}\Omega_l \nabla J_{k,l}^0. \tag{3.27}$$

The expressions (3.27) and (3.26) may be inserted in (3.22) and (3.24) to get:

$$\frac{1}{c}\partial_{\tilde{t}}(E_q^0) - \mathrm{div}\Big(\frac{1}{\tilde{\sigma}_q}\nabla\Big((DE)_q^0\Big)\Big) = -\tilde{\sigma}_q E_q^2,$$
$$\frac{1}{c}\partial_{\tilde{t}} J_{k,l}^0 - \Omega_l \mathrm{div}\Big(\frac{1}{\tilde{\sigma}_k}\Omega_l \nabla J_{k,l}^0\Big) = -\tilde{\sigma}_k J_{k,l}^2.$$

Then E_q^0 and $J_{k,l}^0$ can be replaced by $a\theta_q^4$ and$\mathcal{B}_k(T)$ thanks to (3.14) and (3.16):

$$\frac{1}{c}\partial_{\tilde{t}}\big(a\theta_q^4\big) - \mathrm{div}\Big(\frac{1}{3\tilde{\sigma}_q}\nabla\Big(a\theta_q^4\Big)\Big) = -\tilde{\sigma}_q E_q^2, \tag{3.28}$$
$$\frac{1}{c}\partial_{\tilde{t}}\mathcal{B}_k(T) - \Omega_l \mathrm{div}\Big(\frac{1}{\tilde{\sigma}_k}\Omega_l \nabla \mathcal{B}_k(T)\Big) = -\tilde{\sigma}_k J_{k,l}^2. \tag{3.29}$$

Let us now sum over all l the second equation. Since $\mathcal{B}$ is isotropic and the quadrature formula $\overline{<\bullet>}$ exactly integrates 2nd order polynomials, we get:

$$\frac{4\pi}{c}\partial_{\tilde{t}}\mathcal{B}_k(T) - \mathrm{div}\Big(\frac{4\pi}{3\tilde{\sigma}_k}\nabla\mathcal{B}_k(T)\Big) = -\sum_{l=1}^{L}\tilde{\sigma}_k J_{k,l}^2\omega_l.$$

Hence,

$$\frac{4\pi}{c}\partial_{\tilde{t}}\mathcal{B}_k(T) - \mathrm{div}\Big(\frac{4\pi}{3\tilde{\sigma}_k}\partial_T\mathcal{B}_k(T)\nabla T\Big) = -\sum_{l=1}^{L}\tilde{\sigma}_k J_{k,l}^2\omega_l. \tag{3.30}$$

Finally, summing all the equations (3.28) and (3.30) over all groups and bands:

$$\partial_{\tilde{t}}(aT^4) - \frac{c}{3}\mathrm{div}\Big(< \frac{\partial_T B(T)}{\tilde{\sigma}} > \nabla T\Big) = -\sum_{q=1}^{Q}\tilde{\sigma}_q E_q^2 - \overline{< c\tilde{\sigma}J^2 >}, \tag{3.31}$$

This last equality is true thanks to the narrow band hypothesis that allows us to correctly handle the $\tilde{\sigma}_k$. The last step is performed by adding (3.25) to this last equation:

$$\partial_t(\rho C_v T + aT^4) - \nabla_x\Big(\frac{4acT^3}{3\sigma_R}\nabla T\Big) = 0.$$

□

Theorem 3.2. *If $(\omega_k)_k$ are chosen such that $\alpha(T)$ is a positive, decreasing convex function of T, then the model (3.1) − (3.7) has a total entropy which is locally decreasing:*

$$\partial_t\Big(\frac{<\check{h}>}{c} + \rho C_v\mathcal{U}\Big) + \nabla < \Omega\check{h} > \leq 0, \tag{3.32}$$

where $\mathcal{U} = \mathcal{U}(T)$ is a convex function and $\check{h}$ is given by:

$$\check{h} = \begin{cases} h \text{ if } \nu \in \cup\mathcal{Q}_q \\ h_d \text{ if } \nu \in \cup\overline{\mathcal{Q}}_k \end{cases} \tag{3.33}$$

Remark 3.3. In the above theorem α is Lagrange's multiplier that defines $\mathcal{B}$, see (1.4) and (3.7).

Remark 3.4. The condition on $(\omega_k)_k$ is not restrictive for the applications considered in this paper. Indeed, if the quadrature formula was exact, then $\alpha = 1/T$. Only the quadrature errors may prevent α to be positive, decreasing and convex. Whenever only a few narrow bands are considered, it is easy to find a quadrature formula that allows to conserve these properties.

Proof. First, let us recall that the form of $\mathcal{I}$ given in (2.9) depends on a parameter per group denoted by α_q which is a vector of $\mathbf{R}^{1+d}$ (d is the space dimension). In this proof, we will write $\alpha_q = (\alpha_q^0, \alpha_q^x, \alpha_q^y)^\top$, assuming that $d = 2$ (extensions to other dimensions are straightforward).
We will also use the following notations:

$$A_q = \begin{pmatrix} \alpha_q^x & 0 \\ 0 & \alpha_q^y \end{pmatrix},$$
$$\beta_q = \alpha_q.(1,\ \Omega)^\top.$$

It is to note that by construction of $\mathcal{I}$, β_q is nothing but:

$$\beta_q = \partial_I h(\mathcal{I}_q). \tag{3.34}$$

Let us now multiply (**??**) by α_q^0 and (**??**) by A_q:

$$\begin{aligned} &< \alpha_q^0 \partial_t \mathcal{I} >_q + < c\alpha_q^0 \nabla(\Omega\mathcal{I}) >_q =< c\sigma\alpha_q^0 (B(T) - \mathcal{I}) >_q, \\ &< A_q \Omega \partial_t \mathcal{I} >_q + < cA_q \nabla(\Omega \otimes \Omega\mathcal{I}) >_q =< c\sigma A_q \Omega (B(T) - \mathcal{I}) >_q, \end{aligned}$$

Adding these two equations we get:

$$< \beta_q \partial_t \mathcal{I} >_q + < c\beta_q \Omega \nabla \mathcal{I} >_q =< c\sigma\beta_q (B(T) - \mathcal{I}) >_q .$$

Now, replacing β by its value according to (3.34), this last equation becomes:

$$\partial_t < h(\mathcal{I}) >_q + \nabla < c\Omega h(\mathcal{I}) >_q =< c\sigma \partial_I h(\mathcal{I}) \big(B(T) - \mathcal{I}\big) >_q, \tag{3.35}$$

which can be written as:

$$\begin{aligned} \partial_t < h(\mathcal{I}) >_q + \nabla < c\Omega h(\mathcal{I}) >_q =& < c\sigma \big(\partial_I h(B) - \partial_I h(\mathcal{I})\big) \big(B(T) - \mathcal{I}\big) >_q & (3.36) \\ &+ < c\sigma \partial_I h(B) \big(B(T) - \mathcal{I}\big) >_q, & (3.37) \end{aligned}$$

and since h is a convex function and $\partial_I h(B) = 1/T$, we have:

$$\partial_t < h(\mathcal{I}) >_q + \nabla < c\Omega h(\mathcal{I}) >_q \leq < c\frac{\sigma}{T} \big(B(T) - \mathcal{I}\big) >_q . \tag{3.38}$$

Similarly, multiplying (3.3) by $\partial_I h_d(J_{k,l})$ leads to:

$$\frac{1}{c} \partial_I h_d(J_{k,l}) \partial_t J_{k,l} + \partial_I h_d(J_{k,l}) \Omega_l \nabla_l J_{k,l} = \sigma_k \partial_I h_d(J_{k,l}) \big(\mathcal{B}_k - J_{k,l}\big), \tag{3.39}$$

then:

$$\frac{1}{c} \partial_t \big(h_d(J_{k,l})\big) + \Omega_l \nabla \big(h_d(J_{k,l})\big) = \sigma_k \partial_I h_d(J_{k,l}) \big(\mathcal{B}_k - J_{k,l}\big), \tag{3.40}$$

and using the quadrature formula:

$$\partial_t \overline{< h_d(J_{k,l}) >} + \nabla \overline{< c\Omega_l h_d(J_{k,l}) >} = \overline{< c\sigma_k \partial_I h_d(J_{k,l}) \big(\mathcal{B}_k - J_{k,l}\big) >}. \tag{3.41}$$

Proceeding as above and since $\partial_I h_d(\mathcal{B}) = \alpha$ we have:

$$\partial_t \overline{< h_d(J_{k,l}) >} + \nabla \overline{< c\Omega_l h_d(J_{k,l}) >} \leq \overline{< c\sigma_k \alpha \big(\mathcal{B}_k - J_{k,l}\big) >}. \tag{3.42}$$

Now, let us respectively multiply (3.4) and (3.5) by $1/T$ and α (Lagrange's multiplier that defines $\mathcal{B}$, see (1.4)) to get:

$$\frac{\rho C_v}{T}\partial_t(T-\mathcal{T}) =< c\frac{\sigma}{T}\big(\mathcal{I}-B\big) >, \tag{3.43}$$

$$\rho C_v \alpha \partial_t \mathcal{T} = \overline{< c\sigma\alpha\big(J-\mathcal{B}\big)>}. \tag{3.44}$$

Finally, summing (3.38) over q and adding (3.42), (3.43) and (3.44) leads to:

$$\partial_t\big[\frac{<\check{h}>}{c}+\rho C_v\mathcal{U}\big]+\nabla<\Omega\check{h}>\leq 0, \tag{3.45}$$

$$\mathcal{U}=-\int^T \frac{1}{T}\partial_T(T-\mathcal{T})+\alpha\partial_T\mathcal{T}. \tag{3.46}$$

If α has the required behavior, all four terms inside the integral are positive, decreasing and convex functions of T, then $\mathcal{U}$ is a (negative, decreasing) convex function of T. □

4. Numerical Example

This numerical example intends to emphasize the behavior of the hybrid model. A 1 m^2 wide square domain is considered. Radiative equilibrium at $T_0 = 1000\ K$ is initially assumed. For $t > 0$, four "beams" are enforced at the NW, NE and SE corners as well as at the middle of the bottom of the domain (see figure 4.1 below). Each "beam" has the following profile:

$$I_\nu(\Omega)=\begin{cases} B_\nu(T_b),\ \text{if } |\nu-\nu_b|\leq\Delta\nu,\\ B_\nu(T_0),\ \text{otherwise.}\end{cases}$$

where $T_b = 6000\ K$, $\nu_b = 1.10^{14}\ Hz$ and $\Delta\nu = 0.10^{14}\ Hz$. The simulation is made with two groups and one narrow band. Figure 4.1 shows the predicted radiative intensity integrated over all the frequencies and located at coordinates $(0.5,\ 0.5)$ compared with the same computation carried out with a multigroup-$M1$ model:

As figure 4.1 shows, the hybrid model perfectly deals with this simple test-case: each beam is fully preserved with the correct amplitude and direction. Moreover, since there is only one narrow band, the computation is very cheap (almost immediate for a 100×100 mesh).

On the other hand, the $M1$ model, as any direction-integrated model, cannot properly deal with these four beams of the same amplitude converging. This is due to the fact that it has to cope with only one direction of anisotropy. This is a typical case where moments or flux-limited diffusion models fail to predict a realistic solution.

Even though this test is very simple, it is representative of what can be done for simulating a ICF-type problem where dozens of lasers coming from the same source (and hence having the energy concentrated inside the same frequencies) converge.

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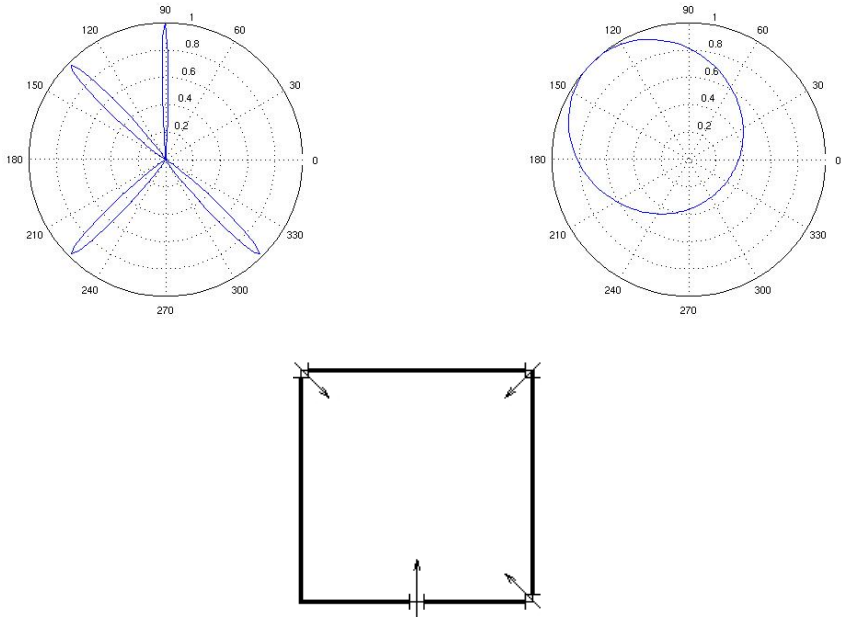

FIGURE 4.1. (above) Directionnal energies given by the hybrid (l.) and $M1$ (r.) models (below) illustration of the considered domain.

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